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Automorphisms of Orthogonal Groups in Characteristic 2

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Let V be a nondefective quadratic space over a field F of characteristic 2. Assume that V has dimension at least ten and that F has more than two elements. Let Δ be one of the groups $O(V), O^+(V), O'(V)$, or $\Omega(V)$ (the full orthogonal group, the rotation group, the spinorial kernel, or the commutator subgroup of $O(V)$, respectively). Then Λ is an automorphism of Δ if and only if $\Lambda(\sigma) = g\sigma g^{-1}$ for all σ in Δ where g is a semilinear automorphism of V that preserves the quadratic structure of V in the sense that $Q(gx) = \alpha Q(x)^u$ for all x in V where Q is the quadratic form, α is some nonzero element of F, and u is the field automorphism of F associated to g.

INTRODUCTION

We are interested in determining the automorphisms of certain subgroups of the orthogonal groups $O(V)$ of a nondefective quadratic space over a field F of characteristic 2. In particular we are interested in describing the automorphisms of $O(V)$, its rotation group $O^+(V)$, its commutator group $\Omega(V)$, and the spinorial kernel $O'(V)$.

The automorphisms of some of these groups have been the subject of investigations by Dieudonné [3], Steinberg [13], Xu [15], and Humphreys [7]. In each of these investigations certain restrictions were placed on the underlying field or the geometry of the underlying space. In [15] for example Xu determines the automorphisms of $\Omega(V)$, $O(V)$, and $O^+(V)$ when V is an isotropic nondefective space and F is perfect. His result generalizes some of the work of Dieudonné; namely his description of the automorphisms of $\Omega(V)$ where V is a nondefective space of dimension at least ten and F is a finite field. The methods employed by Dieudonné and Xu rely heavily on the presence of involutions and the structure of the groups.

In this paper we determine the automorphisms of all of the above mentioned groups in the case where V has dimension at least ten and F has more than two elements. The main results appear as 7.6 and 8.2 in the text. Stated in combined form the main result is as follows; *Let Δ be one of the groups mentioned in the first paragraph; then Λ is an automorphism of Δ if and only if $\Lambda(\sigma) = g\sigma g^{-1}$ for all σ in Δ where g is a semilinear automorphism of V that preserves the quadratic structure of V.*

Our method of approach is the involution free plane rotation method introduced by O'Meara [11]. We first obtain the automorphisms of $\Omega(V)$ and $O'(V)$ (see 7.6) and then use this result to obtain the automorphisms of $O(V)$ and $O^+(V)$ (see 8.2). As in [11] we utilize centralizers and derived groups to show that automorphisms preserve nondefective plane rotations (see 5.14). We then establish a bijection of the nondefective planes of V and use it to establish a bijection of the lines of V. This latter bijection is shown to satisfy the conditions of the Fundamental Theorem of Projective Geometry, and it is then applied to describe the automorphism.

We refer the reader to [3, 5, 9–12] for discussion of the automorphism question and for an extensive bibliography on the subject.

1. Preliminaries

1A. *Assumptions, Conventions, and Some Basics*

Throughout this paper V will be a nondefective quadratic space over a field F of characteristic 2, Q will be used to denote the underlying quadratic form, and $(,)$ will be its associated symmetric bilinear form. We assume familiarity with the theory of quadratic forms in characteristic 2 as treated in [2] and [5]. When possible we employ the notation and terminology of [8], one exception being the use of $(,)$ for the bilinear form. In particular, isotropy is that of [8], not [5]. We also assume familiarity with transvections and involutions as treated in [6], [9], and [10] and with the concepts of fixed and residual spaces of [11]. The results of [11] which are obviously true for characteristic 2 will be used often without statement or proof.

Conventions. (1) Unless otherwise mentioned we assume that $\dim V = 2m = n \geqslant 4$ and $\operatorname{card} F > 2$.

(2) If $\sigma \in O_n(V)$, P and R will denote its fixed and residual spaces, respectively. Similarly with σ_i, P_i, R_i; $\bar{\sigma}_i$, $\bar{P}_i$, $\bar{R}_i$, etc.

(3) If Λ is an automorphism of a subgroup G of O_n and σ is in G,

σ' will be used for $\Lambda\sigma$ and the fixed and residual spaces of σ' will be denoted P' and R', respectively.

If $\sigma \in O_n(V)$ we say that σ is "__" $\Leftrightarrow R$ is "__" (insert any of the words or phrases used to describe the geometry of R twice).

1.1. *If $\sigma \in O_n(V)$ then $\sigma^2 = 1 \Leftrightarrow \sigma$ is totally defective.*

Proof. $\Rightarrow$ See [6, Proposition 3]. $\Leftarrow R \subsetneq R^* = P$. Hence, $\sigma[\sigma x + x] = \sigma x + x \ \ \forall x \in V$. Hence, $\sigma^2 x = x \ \ \forall x \in V$. Q.E.D.

1.2. *If $\sigma \in O_n(V)$ and $\sigma x = \alpha x$ for some $x \in V$, $\alpha \in F$ then $x \in R \cup P$ with $x \in P$ if x is anisotropic.*

1B. *Plane Rotations*

The rotation group O_n^+ may be characterized as those isometries which may be expressed as a product of exactly n orthogonal transvections (where in fact the first, or last, may be arbitrarily chosen). This characterization and Dieudonné's result, [4, Theorem 6] enable us to characterize O_n^+ as those isometries with even residual index. The rotations with residual index 2 are called *plane rotations*. Note that 1.1 shows that a plane rotation σ is nondefective if and only if $\sigma^2 \neq 1$.

The remainder of this subparagraph is devoted to characterizing the plane rotations with given residual space R. As in [11] this characterization depends on the geometry of R. The case R nondefective presents no problems—we state it without proof.

1.3. *Let R be a nondefective plane in V. The set of plane rotations with res space R is $O_2^+(R) \perp 1_P$ (exclude 1).*

The set of plane rotations in O_n' or Ω_n is $\Omega_2(R) \perp 1_P$ (exclude 1).

1.3a. *Let R be any nondefective plane in V. Then there is a σ in Ω_n with res space $\sigma = R$.*

Proof. Express $R = Fx + Fy$ where $(x, y) = 1$ and $Q(x) \cdot Q(y) \neq 0$. Then $\sigma = \tau_x \tau_{Q(y)x+y}$ does the job. Q.E.D.

1.4. *Let σ be a nondegenerate plane rotation. Then $\sigma = \tau_a \tau_b$ where a is any anisotropic vector in R and $R = Fa + Fb$. If σ is defective then $\sigma \notin O_n'$.*

Proof. The nondefective case is easy, so we omit this part of the proof. Assume R is defective (but nondegenerate). Choose W quaternary, nondefective, and containing R. Then $W = (Fa + Fg) \perp (Ff + Fh)$, where $R = Fa \perp Ff$ and $\sigma W = W$. Thus we may assume that $W = V$. Let T

be the fixed space of $\sigma\tau_a$. Then $R \subset T$; hence, T is ternary and $\sigma\tau_a$ is a transvection τ_b, $b \in T$. We must show that $b \in R$. Now $\tau_b = \sigma\tau_a = \sigma\tau_{\sigma(a)} = \sigma\sigma\tau_a\sigma^{-1} = \sigma\tau_b\sigma^{-1} = \tau_{\sigma(b)}$. Thus, $\sigma b = \alpha b$. Apply 1.2. The rest is obvious. Q.E.D.

1.4a. *Every plane rotation in O_n' is either nondefective or degenerate. If V is anisotropic every plane rotation in O_n' is nondefective.*

We complete our characterization of the plane rotations with given residual space by examining the degenerate plane rotations. It will be shown that all such belong to Ω_n and are the $E_{i,w}$ of Eichler (see [14]).

Let i be an isotropic vector in V and let $w \in Fi^*$. For $x \in V$ let $E_{i,w}(x) = x + (x, w)\, i + (x, i)\, w + Q(w)(x, i)\, i$. Then $E_{i,w} \in O_n$ and the following hold for $\alpha \in \dot{F}$ and $w \in F_i^*$:

$$E_{i,w} = 1 \text{ if and only if } w \in Fi;$$
$$E_{i,w}E_{i,u} = E_{i,w+u};$$
$$(E_{i,w})^2 = 1;$$
$$E_{\alpha i,w} = E_{i,\alpha w};$$
$$E_{i,w} = \tau_w\tau_{w+Q(w)i} \text{ if } Q(w) \neq 0;$$
$$\sigma E_{i,w}\sigma^{-1} = E_{\sigma i,\sigma w} \qquad \forall \sigma \in O_n.$$

The residual space of $E_{i,w}$ is $Fi + Fw$. In particular, $E_{i,w}$ is a plane rotation if $w \notin Fi$. In fact $E_{i,w} \in \Omega_n$ and if V is isotropic, Ω_n is the group generated by all such $E_{i,w}$ [14], Lemma 12]. (Recall our convention (1).)

Now let σ be a degenerate plane rotation. Then $R = Fe_1 \perp Fe_2$ with $Q(e_1) = 0$. Clearly σ is totally degenerate if and only if $Q(e_2) = 0$. Extend $\{e_1, e_2\}$ to a symplectic base $\{e_i, f_i\}_{i=1}^m$. Then

$$(e_i, e_j) = (f_i, f_j) = 0, \qquad 1 \leqslant i, j \leqslant m \qquad \text{and} \qquad (e_i, f_j) = \delta_{i,j}$$

(Kronecker delta). We may choose f_1 such that $Q(f_1) = 0$, and if $Q(e_2) = 0$ we may choose f_2 with $Q(f_2) = 0$. An easy computation shows that

$$\sigma(e_i) = e_i \qquad 1 \leqslant i \leqslant m,$$
$$\sigma(f_i) = f_i \qquad i > 2,$$
$$\sigma(f_1) = \alpha e_1 + f_1 + \lambda e_2,$$
$$\sigma(f_2) = \lambda e_1 + \beta e_2 + f_2,$$

where $\lambda \neq 0$, $\alpha + \lambda^2 Q(e_2) = 0$ and $\beta^2 + \beta = 0$.

If $\beta = 1$ the Dickson invariant of σ is 1 which is not possible if σ is a rotation. Thus $\beta = 0$.

Now consider $E_{e_1,\lambda e_2}$. Using the above base it is easy to show that $E_{e_1,\lambda e_2} = \sigma$. We summarize.

1.5. *Let R be a degenerate plane in V with $R = Fi \perp Fw$, $Q(i) = 0$. Then the set of plane rotations with residual space R is $\{E_{i,\lambda w} \mid \lambda \in \dot{F}\}$. Each $E_{i,w} \in \Omega_n$ and if V is isotropic Ω_n is generated by all such maps.*

1.5a. *If $n \geqslant 6$ and V is isotropic or if $n = 4$ and the Witt index is 1 then Ω_n is generated by all $E_{i,w}$ where $Q(w) \neq 0$.*

Proof. Recall $\sigma E_{i,w}\sigma^{-1} = E_{\sigma i,\sigma w}$. Now use the simplicity of Ω_n and 1.5. Q.E.D.

1.6. *Suppose $E_{i,w} \in \Omega_n$. Then there is an $E_{j,v} \in \Omega_n$ such that $E_{i,w}E_{j,v}$ is a nondefective plane rotation in Ω_n.*

Proof. Choose j isotropic with $(i, j) = 1$ and $(j, w) = 0$. Define σ by $\sigma i = \alpha i$, $\sigma j = \alpha^{-1} j$, $\sigma \mid (Fi + Fj)^* = 1$, where $\alpha \in \dot{F}^2 - \{1\}$. Then Fi is the intersection of the residual spaces of σ and $E_{i,w}$. Since $E_{i,w}\sigma$ is a rotation in Ω_n it is a plane rotation. An easy computation shows that it is defective. Now apply 1.4a and 1.5. Q.E.D.

1.7. *Let σ be a plane rotation in O_n^+. Then there are vectors a, b, c and d with $R = Fa + Fb = Fc + Fd$ and $\sigma x = x + (x, c)\, a + (x, d)\, b$, $\forall x \in V$.*

Proof. If σ is nondegenerate express R and σ as in 1.4. If $(a, b) = 0$ let $c = a/Q(a)$, $d = b/Q(b)$. If $(a, b) \neq 0$ assume it is 1 and choose $c = Q(b)\, a + b/Q(a)\, Q(b)$, $d = b/Q(b)$.

If σ is degenerate then $\sigma = E_{i,w}$ where $R = Fi \perp Fw$. Let $a = i$, $b = w$, $c = w + Q(w)\, i$ and $d = i$. Q.E.D.

1C. *Permutability*

The proofs of the following results may be lifted from the analogous propositions in [11] (namely 1.11, 1.19, 1.20, and 1.21).

1.8. *If σ_1 and σ_2 are in O_n with σ_1 nondefective*

(i) $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow \sigma_2 R_1 = R_1$ *and* $\sigma_2 \mid R_1$ *permutes with* $\sigma_1 \mid R_1$;

(ii) $(R_1, R_2) = 0 \Rightarrow \sigma_1\sigma_2 = \sigma_2\sigma_1$;

(iii) $\sigma_1\sigma_2 = \sigma_2\sigma_1$ *and* $R_1 \cap R_2 = 0 \Rightarrow (R_1, R_2) = 0$.

1.9. *Let σ be a nondefective plane rotation, and let $\Sigma \in O_n$ then $\Sigma\sigma = \sigma\Sigma \Leftrightarrow \Sigma R = R$ and $\Sigma \mid R \in O_2^+(R)$.*

1.10. *Let σ_1 and σ_2 be plane rotations with one of them nondefective. Then $\sigma_1\sigma_2 = \sigma_2\sigma_1 \Leftrightarrow R_1 = R_2$ or $(R_1, R_2) = 0$.*

1.11. *Let σ_1 and σ_2 be plane rotations then $\sigma_1\sigma_2$ is a plane rotation $\Leftrightarrow R_1 \cap R_2 \neq 0$ and $\sigma_1 \neq \sigma_2^{-1}$.*

2. The Derived Groups D^kO_n

We denote the commutator subgroup of a group G by DG. The groups D^kG are defined inductively with $D^0G = G$ and $D^{k+1}G = D(D^k(G))$. Then $D^kG \,\Delta\, G$ for all k. If $G = O_n$ then $DG = \Omega_n$.

2.1. *If $V = U \perp U^*$ is a nontrivial splitting then*

$$D^kO(U) \perp D^kO(U^*) \subseteq D^kO(V).$$

2.2. *If V is isotropic then $D^kO = \Omega$ for $k \geqslant 1$.*

Proof. [14, Lemma 13].

2.3. *Suppose U is nondefective $0 \subset U \subset V$ then $\exists\sigma \in O(V)$ with $\sigma U \not\subseteq U \cup U^*$ and $\sigma U \cap U \neq 0$. In fact we may choose $\sigma = \tau_{y+w}$ where y is any element of U and w is any nonzero element of U^* with $Q(y) \neq Q(w)$.*

2.4. *$D^kO(V)$ contains a nondefective plane rotation for all $k \geqslant 0$.*

Proof. Choose x and y in V with $(x, y) = 1$ and $Q(x)\,Q(y) \neq 0$. Then $R = Fx + Fy$ is nondefective. Let $\sigma = \tau_x\tau_y\tau_x\tau_y$. Then $\sigma \neq 1$, $\sigma \in \Omega$ and res space $\sigma = R$. This allows us to assume that $k \geqslant 2$ and V is anisotropic. We induce on k. Let $\sigma \in D^kO$ with σ a nondefective plane rotation. Then $\sigma \in \Omega$ and $\sigma = \tau_x\tau_y$ for some choice of x and y in $R = Fx + Fy$. As in 2.3 choose w in R^* with $Q(w) + Q(y) \neq 1$. Let $\tau = \tau_{y+w}$ and consider $\tau\sigma\tau^{-1} = \Sigma$. Then Σ is a nondefective plane rotation in D^kO with res space τR. But $\tau R \cap R \neq 0$ and $(\tau R, R) \neq 0$ as in 2.3, and, hence, $\sigma\Sigma = \Sigma\sigma$ by 1.10. A direct computation (utilizing the condition $Q(y) + Q(w) \neq 1$) will show that $\Sigma R \cap R \neq 0$. Hence, $\sigma\Sigma\sigma^{-1}\Sigma^{-1}$ is a plane rotation in $D^{k+1}O$ by 1.11. It is nondefective by 1.4a.

Q.E.D.

2.5. *Let $k \geqslant 0$. Then there are two nondefective plane rotations in D^kO whose residual spaces intersect in an anisotropic line.*

Proof. Let $k \geqslant 0$ be given. Choose σ as guaranteed by 2.4. Express $R = Fx + Fy$ with $Q(y) \neq 0$ and $(x, y) = 1$. Choose τ as in 2.3, and let $\Sigma = \tau\sigma\tau^{-1}$. Then Σ and σ are the desired elements. Q.E.D.

2.6. *Let $\sigma \in O$.*

(i) *If σ leaves all lines in V fixed then $\sigma = 1$.*

(ii) *If V is isotropic and σ leaves all isotropic lines fixed then $\sigma = 1$.*

(iii) *Let $x \in V$ and let $\mathscr{L} = \{Fy \mid Q(y) = Q(x)\}$. If σ leaves all lines in $\mathscr{L}$ fixed then $\sigma = 1$.*

Proof. (i) and (ii) are straightforward. (iii) follows from (ii) if $Q(x) = 0$ so assume $Q(x) \neq 0$. Then $\sigma x = x$. Suppose $y \in V$ with $(x, y) = 1$ and $Q(y) \neq 0$. Then $Q(x + 1/Q(y)\,y) = Q(x)$, and, hence, σ leaves $Fx + Fy$ pointwise fixed. Now let z be any vector in V, $z \neq 0$. If $(x, y) \neq 0$ the above shows that $\sigma z = z$. So assume $(x, z) = 0$. Then $(x, y + z) = 1$ where y is chosen as above. Then $\sigma(y + z) = y + z$ as above and $\sigma z = \sigma(y + y + z) = \sigma(y) + \sigma(y + z) = y + y + z = z$. Q.E.D.

2.7. *Let $k \geqslant 0$. The centralizer of the set of nondefective plane rotations in D^kO is 1.*

Proof. Choose σ_1 and σ_2 as in 2.5 with $R_1 \cap R_2 = Fx$. Let φ be in the above named centralizer and let y be such that $Q(y) = Q(x)$. Choose τ in O with $\tau(x) = y$. Then $\tau\sigma_i\tau^{-1}$, $i = 1, 2$ is a nondefective plane rotation in D^kO. Hence, φ permutes with $\tau\sigma_i\tau^{-1}$ and so $\varphi(\tau R_i) = \tau R_i$, $i = 1, 2$. Thus, $\varphi(Fy) = Fy$. Apply 2.6 (iii). Q.E.D.

2.7a. *The center of D^kO is 1 for $k \geqslant 0$.*

2.8. *Suppose U is a nondefective subspace with $0 \subset U \subset V$. Let l be any line in V. Then there is a σ in Ω with $\sigma l \not\subseteq U \cup U^*$.*

Proof. We may assume $l \subset U$. Choose $x \in l$ and y in U with $(x, y) = 1$. If x is isotropic choose y isotropic as well. In this case let $\sigma = E_{y,w}$ where $w \in U^*$, $w \neq 0$. Otherwise let $\sigma = \tau_{y+w}\tau_x\tau_{y+w}\tau_x$ where $w \in U^*$, $w \neq 0$, $Q(w) \neq Q(y)$. Q.E.D.

2.9. *Let U be a nondefective subspace of V with $0 \subset U \subset V$. Let $k \geqslant 0$ then $\exists \sigma \in D^kO \ni \sigma U \neq U$.*

Proof. First, assume that V is quaternary. Then U is a plane. If V is isotropic apply 2.2 and 2.8. Thus we may assume that V is anisotropic and $k \geqslant 1$. Select σ_1 in $O_2^+(U)$ with $\sigma_1 \neq 1$. Let $\sigma = \sigma_1 \perp 1_{U^*}$. Using 2.7 we choose Σ a nondefective plane rotation in D^kO with $\sigma\Sigma \neq \Sigma\sigma$. If $\Sigma U = U$ then $\Sigma \mid U \notin O_2^+(U)$ by 1.9. Thus $\Sigma \mid U^* \notin O_2^+(U^*)$ and we have $\Sigma = \tau_x\tau_y$ where $x \in U$, $y \in U^*$. This contradicts the nondefectiveness of Σ. Hence, $\Sigma U \neq U$ as desired.

Now assume V arbitrary. Select nondefective planes $P_1 \subseteq U$, $P_2 \subseteq U^*$ and apply the above argument to $P_1 \perp P_2$. Q.E.D.

2.10. *Let U be a nondefective subspace with $0 \subset U \subset V$. Let $k \geqslant 0$ and let l be a line in V. Then $\exists \sigma \in D^kO \ni \sigma l \nsubseteq U \cup U^*$.*

Proof. We may assume $l \subseteq U$. If V is isotropic apply 2.2 and 2.8. If V is anisotropic it is enough to show that $\sigma l \nsubseteq U$ for some σ in D^kO. The general case follows from the case V quaternary as in 2.10. Hence, we assume that U is a plane. Suppose we produce σ_1 in D^kO with $\sigma_1 l \neq l$. If $\sigma_1 l \nsubseteq U$ choose $\sigma = \sigma_1$. If $\sigma_1 l \subseteq U$ then l and $\sigma_1 l$ span U. Choose Σ in D^kO with $\Sigma U \neq U$ as in 2.9. Then we have $\Sigma\sigma_1 l \nsubseteq U$ or $\Sigma l \nsubseteq U$, and we choose $\sigma = \Sigma\sigma_1$ or $\sigma = \Sigma$ accordingly.

We now produce such a σ_1. Choose x and y in U with $Fx = l$, $(x, y) = 1$. We may assume $Q(x) = 1$. By 2.9 we may choose $\bar{\sigma}$ in D^kO with $\bar{\sigma}U \neq U$. If $\bar{\sigma}x \neq x$ set $\bar{\sigma} = \sigma_1$. Otherwise let $\sigma_1 = \tau_y\bar{\sigma}\tau_y$. Q.E.D.

2.11. (Irreducibility of D^kO). *Let U be any proper subspace of V. Let $k \geqslant 0$. Then $\exists \sigma \in D^kO \ni \sigma U \neq U$.*

Proof. If U is nondefective apply 2.9. If U is defective it is enough to produce σ in D^kO with $\sigma(U \cap U^*) \neq U \cap U^*$. This allows us to assume that U is totally defective.

If V is anisotropic choose x in U and y in V with $(x, y) = 1$. We may assume that $Q(y) = 1$. Let $P = Fx + Fy$. By 2.9 we may choose Σ in D^kO with $\Sigma P \neq P$. If $\Sigma U \neq U$ we are done. Assume that $\Sigma U = U$. Since $U \subseteq U^*$ we have $\Sigma x = x$ by the anisotropy of V. Since $\Sigma P \neq P$ we have $\Sigma y \notin P$. Consider $\tau_y\Sigma\tau_y$. This isometry is in D^kO and $\tau_y\Sigma\tau_y(x) \neq x$. If $\tau_y\Sigma\tau_y(x) \in U$ the total defectiveness of U contradicts the anisotropy of V.

Now assume that V is isotropic. By 2.2 it is enough to produce a σ in Ω with the desired property. If U contains an isotropic vector x choose y isotropic with $(x, y) = 1$ and choose w in $[Fx + Fy]^*$ with $Q(w) \neq 0$. Then $y \notin U$ and $E_{y,w}(x) \notin U$. If U contains no isotropic vectors choose x

in U and y in V with $(x, y) = 1$. We may assume $Q(y) = 1$. Then $\tau_x\tau_y\tau_x\tau_y(x) = y \notin U$. Q.E.D.

2.12. *Let $\sigma \in O$ with $\sigma \neq 1$ and assume there is an a in P with $Q(a) \neq O$. If $k \geqslant 0$ $\exists\, u \in D^kO \ni \sigma u \sigma^{-1} u^{-1} = \tau_c\tau_b$ where $Q(c) = Q(b) = Q(a)$.*

Proof. Choose Σ in D^kO with $\Sigma a \notin R \cup P$ (if σ is nondefective use 2.10; if σ is defective observe that $R + P \subset V$ but that the space spanned by $\{\Sigma a \mid \Sigma \in D^kO\} = V$). Let $a = b$. Set $c = \sigma b$. Then $Fc \neq Fb$. Choose $u = \Sigma\tau_a\Sigma^{-1}\tau_a$. Q.E.D.

2.13. *Let $k \geqslant 0$ and $n \geqslant 6$ or $n = 4$ when the Witt index is 1. Suppose that H is a nontrivial subgroup of O which is invariant under conjugation by elements of D^kO. Then H contains a nondefective plane rotation and, if V is isotropic, $H \supseteq \Omega$.*

Proof. Assume that V is isotropic. Note that in this case Ω is simple (see [5, pp. 68–69]) and, hence, $H \cap \Omega = 1$ or Ω. If $H \cap \Omega = 1$ we choose σ_1 in H and Σ_1 in D^kO. Then $\sigma_1\Sigma_1\sigma_1^{-1} = \Sigma_2 \in D^kO$ and $\Sigma_1^{-1}\Sigma_2 \in D^kO$. But $\Sigma_1^{-1}\Sigma_2$ also is in H since H withstands conjugation by elements of D^kO. Thus $\Sigma_1 = \Sigma_2$ or $\sigma_1\Sigma_1 = \Sigma_1\sigma_1$. Now apply 2.7 to conclude $H = 1$, a contradiction. Thus $H \cap \Omega = \Omega$ so $H \supseteq \Omega$, and we may apply 2.4.

Now assume that V is anisotropic. Choose σ_1 in H with $\sigma_1 \neq 1$. Apply 2.7 to produce a nondefective plane rotation Σ in D^kO with $\sigma_1\Sigma \neq \Sigma\sigma_1$. Thus $\sigma_1\Sigma\sigma_1^{-1}\Sigma^{-1}$ is a nontrivial element of H. The fixed spaces of $\sigma_1\Sigma\sigma_1^{-1}$ and Σ^{-1} are of dimension $n - 2$ and, hence, intersect nontrivially since $n \geqslant 6$. Thus the fixed space of $\sigma_1\Sigma\sigma_1^{-1}\Sigma^{-1}$ contains an anisotropic vector a and by 2.12 $\sigma_1\Sigma\sigma_1^{-1}\Sigma^{-1} = \tau_b\tau_c$, where $Q(b) = Q(c) = Q(a)$. Since V is anisotropic $(b, c) \neq 0$, and, hence, $\tau_b\tau_c$ is nondefective. Q.E.D.

2.14. *Let n and k be as in 2.13, and let $\sigma_i \in O$, $\sigma_i \neq 1$, $i = 1, 2$. Then $\exists \Sigma_1$ in $D^k \ni \Sigma_1\sigma_1\Sigma_1^{-1}$ and σ_2 do not permute.*

Proof. We may assume $k \geqslant 1$. It is enough to produce Σ_1 and Σ_2 in D^kO such that $\Sigma_1\sigma_1\Sigma_1^{-1}$ and $\Sigma_2\sigma_2\Sigma_2^{-1}$ do not permute.

Let $G_i = \{\prod_{\text{fin}} (\Sigma_i\sigma_i^{\pm 1}\Sigma_i^{-1}) \mid \Sigma_i \in D^kO \;\; i = 1, 2\}$. We are done if we produce $\bar{\sigma}_i$ in G_i such that $\bar{\sigma}_1\bar{\sigma}_2 \neq \bar{\sigma}_2\bar{\sigma}_1$.

Note that each G_i may be the "H" of 2.13. Select $\bar{\sigma}_i$ in G_i such that $\bar{\sigma}_i$ is a nondefective plane rotation, $i = 1, 2$. If $\bar{\sigma}_1\bar{\sigma}_2 = \bar{\sigma}_2\bar{\sigma}_1$ apply 2.10 with $U = \bar{R}$, to obtain Σ in D^kO with $\Sigma\bar{R}_1 \neq \bar{R}_2$ and $\Sigma\bar{R}_1 \not\subset \bar{P}_2$. Then $\Sigma\bar{\sigma}_1\Sigma^{-1}$ and $\bar{\sigma}_2$ are the desired elements. If $\bar{\sigma}_1\bar{\sigma}_2 \neq \bar{\sigma}_2\bar{\sigma}_1$, $\bar{\sigma}_1$, and $\bar{\sigma}_2$ are the desired elements. Q.E.D.

3. Centralizers

Let Δ denote either of the groups Ω or O'. If $X \subseteq \Delta$ we use $C(X)$ to denote $\{\Sigma \in \Delta \mid \Sigma\sigma = \sigma\Sigma \ \forall \sigma \in X\}$. Then $C(X)$ is a subgroup of Δ, and we have

$$X_1 \subseteq X_2 \Rightarrow C(X_2) \subseteq C(X_1)$$
$$X \subseteq CC(X)$$
$$CCC(X) = C(X)$$

If Λ is an automorphism of Δ then $\Lambda C(X) = C(\Lambda X)$.

3.1. *Let σ be a nondefective element in Δ then for any $k \geqslant 0$*

$$1_R \perp D^{k+1}O(P) \subseteq D^kC(\sigma) \subseteq D^kO(R) \perp D^kO(P).$$

Proof. As in 1.36 of [11]. Q.E.D.

3.2. *Let $n \geqslant 6$ and let σ be a nondefective plane rotation in Δ. Then $CC(\sigma) = \Omega_2(R) \perp 1_P$.*

Proof. Let $\Sigma \in CC(\sigma)$. Then $\Sigma \in \Delta$, $\Sigma\sigma = \sigma\Sigma$ and $\Sigma = \Sigma \mid R \perp \Sigma \mid P$. Applying 3.1 with $k = 0$ yields $1_R \perp \Omega(P) \subseteq C(\sigma)$. Hence, Σ permutes with every element of the form $1_R \perp \tau$ where $\tau \in \Omega(P)$. Thus, $\Sigma \mid P$ permutes with every element of $\Omega(P)$, and, hence, by 2.7 $\Sigma \mid P = 1_P$. Clearly, $\Sigma \mid R \in \Omega_2(R)$.

Now consider $\bar{\tau} = \tau \perp 1_P$ where $\tau \in \Omega_2(R)$, $\tau \neq 1$. Then $\bar{\tau}$ is a nondefective plane rotation with residual space R. Let $u \in C(\sigma)$. Then $u \mid R \in O_2^+(R)$ by 1.9 and $\bar{\tau}u = u\bar{\tau}$. Hence, $\bar{\tau} \in CC(\sigma)$. Q.E.D.

3.3. *Let $n \geqslant 6$, and let σ be a defective plane rotation in Δ. Thus, σ is degenerate and $\sigma = E_{i,w}$ with $R = Fi \perp Fw$. Then $CC(\sigma) = \{E_{i,\lambda w} \mid \lambda \in F\}$.*

Proof. Express $V = (Fi + Fj) \perp (Fw + Fk) \perp W$ where $(i, j) = 1 = (w, k)$. Note that W is nondefective and $P = R \perp W$.

Choose anisotropic vectors x and y in W with $(x, y) = 1$ and set $\Pi_1 = Fx + F(y + i)$, $\Pi_2 = Fx + F(y + w)$. Note that Π_1 and Π_2 are nondefective planes in P and $\Pi_1 \cap \Pi_2 = Fx$. In the following understand $i = 1, 2$.

Select $\bar{\sigma}_i \in \Omega_2(\Pi_i)$, $\bar{\sigma}_i \neq 1$, and set $\sigma_i = \bar{\sigma}_i \perp 1_{\Pi_i^*}$. Then σ_i is a nondefective plane rotation in Δ with residual space Π_i. Since $\Pi_i \subset P$ we have $\sigma \mid \Pi_i = 1$, and, hence, $\sigma_i \in C(\sigma)$ by 1.10. Now let $\Sigma \in CC(\sigma)$. Thus, $\Sigma\sigma_i = \sigma_i\Sigma$ and $\Sigma \in \Delta$. By 1.9 $\Sigma \mid \Pi_i \in O_2^+(\Pi_i)$. Since $\Pi_1 \cap \Pi_2 = Fx$, $\Sigma x = x$. But x was an arbitrary anisotropic vector in W, and, hence,

Σ leaves fixed all anisotropic vectors in W. By 2.6 (iii) this means that $\Sigma \mid W = 1$. But $\Sigma \mid \Pi_i \in O_2^+(\Pi_i)$. Hence, Σ leaves $y + i$ and $y + w$ fixed. Since Σ leaves y fixed this means Σ leaves i and w fixed, and, hence, $\Sigma \mid P = 1_P$. The residual space of Σ is even dimensional and contained in R by the above. Thus, $\Sigma = E_{i,\lambda w}$ $\lambda \in F$ by 1.5.

Now let $E_{i,\lambda w}$ $\lambda \in F$ be given. We show that $E_{i,\lambda w} \in CC(E_{i,w})$. If $\lambda = 0$ it is clear.

Let $T \in C(E_{i,w})$. Then $TE_{i,w}T^{-1} = E_{i,w}$. Hence, $E_{Ti,Tw} = E_{i,w}$ and $TR = R$. If $Q(w) \neq 0$ then $Ti = \alpha i$ and $Tw = \beta i + w$. We claim $\alpha = 1$. Now $E_{i,w} = E_{Ti,Tw} = E_{\alpha i,\beta i+w} = E_{\alpha i,\beta i}\ E_{\alpha i,w} = E_{\alpha i,w}$. Choose x with $(x, i) = 0$ and $(x, w) = 1$. Then $E_{i,w}x = E_{\alpha i,w}x$ implies $i = \alpha i$.

Now $E_{Ti,T(\lambda w)} = E_{i,\lambda\beta i+\lambda w} = E_{i,\lambda w}$. Thus, $E_{i,\lambda w}T = TE_{i,\lambda w}$ or $E_{i,\lambda w} \in CC(E_{i,w})$ as desired.

If $Q(w) = 0$ the relation $E_{Ti,Tw} = E_{i,w}$ means that $(x, i)w + (x, w)i = (x, Ti)\,Tw + (x, Tw)\,Ti$ for all x. Multiply both sides by λ and add x to get $E_{i,\lambda w} = E_{Ti,T(\lambda w)}$. Q.E.D.

3.4. *If $n \geqslant 6$ and σ is a plane rotation then $CC(\sigma)$ consists exactly of those elements in Δ with residual space R along with* 1.

Proof. Apply 1.3 and 3.2 if σ is nondefective and 1.5 and 3.3 if σ is defective. Q.E.D.

3.5. *Let σ_1 and σ_2 be plane rotations in Δ with both nondefective if $n = 4$. Then $R_1 = R_2 \Rightarrow C(\sigma_1) = C(\sigma_2)$.*

Proof. If $n = 4$ apply 1.9. If $n \geqslant 6$ apply 3.2 or 3.3. Q.E.D.

3.6. *Let $n \geqslant 6$, and let σ_1 be a plane rotation in Δ and let σ_2 be any element in Δ $\sigma_2 \neq 1$. Then $C(\sigma_1) \subseteq C(\sigma_2) \Rightarrow \sigma_2$ is a plane rotation and $R_2 = R_1$. Hence σ_2 is nondefective if and only if σ_1 is.*

Proof. $C(\sigma_1) \subseteq C(\sigma_2) \Rightarrow CC(\sigma_2) \subseteq CC(\sigma_1)$. Hence, $\sigma_2 \in CC(\sigma_1)$. Apply 3.2 or 3.3. Q.E.D.

3.7. *Assumptions as in* 3.6. *The following are equivalent*

(i) $C(\sigma_1) \subseteq C(\sigma_2)$;

(ii) $P_2 = P_1$;

(iii) $\sigma_2 \mid P_1 = 1_{P_1}$.

In particular if any of these hold, σ_i as given, then σ_1 and σ_2 are both plane rotations with σ_1 nondefective if and only if σ_2 is nondefective.

Proof. (i) $\Rightarrow$ (ii) follows from 3.6.

(ii) $\Rightarrow$ (iii) is obvious.

(iii) $\Rightarrow$ (i) Now $\dim P_1 = n - 2$. Hence, $\sigma_2 \mid P_1 = 1$ implies that $P_2 = P_1$, P_2 is a hyperplane or $P_2 = V$. But $P_2 = V$ means $\sigma_2 = 1$ and P_2 a hyperplane means $\sigma_2 \notin O^+$. Q.E.D.

4. The Groups $DCD^kC(\sigma)$

4.1. *Let $n \geqslant 6$ and $k \geqslant 2$. Let σ be a nondefective plane rotation in Δ. Then $\Sigma P = P$ for all Σ in $D^kC(\sigma)$ and $X_k = \{(\Sigma \mid P) \mid \Sigma \in D^kC(\sigma)\}$ is a subgroup of $O^+(P)$. In fact*

(i) $D^{k+1}O(P) \subseteq X_k \subseteq D^kO(P)$;

(ii) $D^kC(\sigma) = 1_R \perp X_k$;

(iii) $CD^kC(\sigma) = \Omega_2(R) \perp 1_P$. *In particular $CD^kC(\sigma)$ is abelian if $k \geqslant 2$.*

Proof. If $\Sigma \in D^kC(\sigma)$ then $\Sigma\sigma = \sigma\Sigma$. Hence, $\Sigma R = R$, $\Sigma P = P$, $\Sigma \mid R \in O_2^+(R)$ and $\Sigma \mid P \in O^+(P)$. The subgroup claim is obvious:

(i) follows from 3.1 and commutativity of O_2^+.

(ii) follows from 3.1 and the definition of X_k.

(iii) If $\bar{\sigma} \in \Omega_2(R) \perp 1_P$ then $\bar{\sigma} \in \Delta$. By (ii) and 1.8 (ii) $\bar{\sigma} \in CD^kC(\sigma)$. Now let $\bar{\sigma} \in CD^kC(\sigma)$ then $\bar{\sigma} \in \Delta$ and $\bar{\sigma}\Sigma = \Sigma\bar{\sigma}$ for all Σ in $D^kC(\sigma) = 1_R \perp X_k$. By (i) $\bar{\sigma}$ permutes with every element of $1_R \perp D^{k+1}O(P)$. Hence, $\bar{\sigma} \mid P$ permutes with every element of $D^{k+1}O(P)$. Thus, $\bar{\sigma} \mid P = 1_P$ by 2.7. Clearly $\bar{\sigma} \mid R \in \Omega_2(R)$. The last remark follows since $\Omega_2 \subseteq O_2^+$ and O_2^+ is abelian. Q.E.D.

4.2. *Let $n \geqslant 8$ and $k \geqslant 2$. Let σ be a nondefective plane rotation in Δ, and let $\sigma_i \in D^kC(\sigma)$ $\sigma_i \neq 1$ $i = 1, 2$. Then $\exists \Sigma_1 \in D^kC(\sigma) \ni \Sigma_1\sigma_1\Sigma^{-1}$ and σ_2 do not permute.*

Proof. Apply 4.1 and 2.14. Q.E.D.

4.3. *Let $k \geqslant 0$ and let σ be a nondefective element in Δ. Let $\Sigma \in D^{k+1}C(\sigma)$. Thus, $\Sigma P = P$, $\Sigma R = R$. Then $\Sigma \mid P \perp 1_R$ and $1_P \perp \Sigma \mid R$ are in $D^kC(\sigma)$.*

Proof. As in 2.3 of [11]. Q.E.D.

4.4. *Let $k \geqslant 0$ and let σ be a nondefective element in Δ with* $\dim P \geqslant 4$ *and* $\dim R \geqslant 4$. *Assume also that $DCD^{k+2}C(\sigma)$ is abelian. Then $D^kC(\sigma)$ contains noncentral elements Φ and Ψ such that*

(i) $\Phi \mid P = 1_P \, \Psi \mid R = 1_R$,

(ii) $\Sigma\Phi\Sigma^{-1}$ *and Ψ permute for all Σ in $D^kC(\sigma)$.*

Proof. As in 2.4 of [11]. Q.E.D.

4.5. *Let $n \geqslant 8$, and let σ be a nondefective plane rotation in Δ. Let $\Sigma \in C(\sigma)$ with $\Sigma \neq 1$ and $C(\Sigma) \neq C(\sigma)$. Then $\exists\, T \in C(\sigma) \ni T\Sigma T^{-1}$ and Σ do not permute.*

Proof. $\Sigma \mid R \in O_2^+(R)$ by 1.9. Apply 2.14 to produce T_0 in $\Omega(P)$ with $T_0(\Sigma \mid P)\, T_0^{-1}$ and $\Sigma \mid P$ nonpermuting. Let $T = 1_R \perp T_0$. Q.E.D.

4.6. *Let σ be a defective rotation in Δ which is nondegenerate and not totally defective. Then $C(\sigma) \subset C(\sigma^2)$.*

Proof. Note the assumptions imply that $n \geqslant 6$. It is enough to produce T in $C(\sigma^2)$ but not in $C(\sigma)$. The assumptions on σ imply that R has a nontrivial splitting $R = R_1 \perp (R \cap P)$ where R_1 is nondefective and $R \cap P$ is anisotropic and totally defective. An easy computation shows that $\sigma - 1$ restricted to R_1 is injective, that the residual space of σ^2 is $\bar{R} = (\sigma - 1)\, R_1$ and $R = \bar{R} \perp (R \cap P)$. Express $V = \bar{R} \perp \bar{R}^*$ and choose x in $R \cap P$ and y in $\bar{R}^*$ with $(x, y) = 1$. Let $\Pi = Fx + Fy$. Then $\Pi \subset \bar{R}^*$. Choose T in $\Omega_2(\Pi)$ with $T \neq 1$. Then $T \in C(\sigma^2)$ by 1.8 (ii). However, $Tx \notin Fx$ since x is anisotropic, and, hence, $Tx \notin R \cap P$. Thus, $T \notin C(\sigma)$. Q.E.D.

4.7. *Let σ be an element in Δ which is degenerate but not totally defective. Then $\exists \Sigma \in C(\sigma)$ with $\Sigma \neq 1$ and $C(\Sigma) \neq C(\sigma) \ni T\Sigma T^{-1}$ and Σ permute $\forall T \in C(\sigma)$.*

Proof. Note that the conditions on σ imply that $n \geqslant 6$ and that σ is not a plane rotation.

As in 4.6 R has a nontrivial splitting $R = R_1 \perp (R \cap P)$ but in this case $R \cap P$ has isotropic vectors. Let i be such a vector and choose w in $R \cap P$ with $w \notin Fi$. Let $\Sigma = E_{i,w}$. Then $\sigma E_{i,w}\sigma^{-1} = E_{i,w}$. Then $C(E_{i,w}) \neq C(\sigma)$ by 3.3. Let $T \in C(\sigma)$. Then $TE_{i,w}T^{-1} = E_{Ti,Tw}$, $TR = R$ and $T(R \cap P) = R \cap P$. Hence, $Fi \perp Fw \subseteq [F(Ti) \perp F(Tw)]^*$. Now apply 4.1 of [10]. Q.E.D.

4.8. *Let σ be in Δ with σ defective but not totally. Then $\exists \Sigma \in C(\sigma)$ with $\Sigma \neq 1$, $C(\Sigma) \neq C(\sigma)$ such that Σ and $T\Sigma T^{-1}$ permute for all T in $C(\sigma)$.*

Proof. Note again that $n \geqslant 6$ and σ is not a plane rotation.

If σ is degenerate apply 4.7. If σ is not degenerate let $\Sigma = \sigma^2$ and apply 4.6. Q.E.D.

4.9. *Let $n \geqslant 8$ and let Λ be an automorphism of Δ. If σ is a nondefective plane rotation in Δ then $\Lambda\sigma$ is nondefective.*

Proof. $\Lambda\sigma$ is not totally defective since σ is not an involution (apply 1.1.). Apply 4.8 and 4.5. Q.E.D.

4.10. *If $n \geqslant 6$ and σ is a nondefective plane rotation then* res σ' *is* 2, $n - 2$ *or* n.

Proof. We may assume that $n \geqslant 8$. Hence, σ' is nondefective by 4.9. If $V = R' \perp P'$ with dim $R' \geqslant 4$ and dim $P' \geqslant 4$ then apply 4.4 to σ' and 4.2 to σ with $k = 2$ to get a contradiction. Q.E.D.

5. All Nondefective Planes Behave

We say that a plane R *behaves* under Λ if there is a σ with residual space R and res $\sigma' = 2$. Note (1) if R is a nondefective plane that behaves under Λ and $\sigma_i \in \Delta$ $i = 1, 2$ with res space $\sigma_i = R$ then res space $\sigma_1{}' =$ res space $\sigma_2{}'$ (apply 3.2). (2) The plane R behaves under Λ if and only if R' behaves under Λ^{-1}.

5.1. *Let R be a nondefective plane. Then $\exists \Sigma \in \Omega \ni \Sigma R \cap R$ is an anisotropic line.*

Proof. Select z in R^* with $z \neq 0$. Choose y in R with $Q(y) \neq 0$ and $Q(y) \neq Q(z)$. We may assume that $Q(y + z) = 1$. Choose x in R with $(x, y) = 1$ and $Q(x) \cdot Q(y) \neq 0$. Then $\Sigma = \tau_x \tau_{y+z} \tau_x \tau_{y+z}$ will do the job. Q.E.D.

5.2. *Let V be isotropic with* res $E'_{i,w} < n/2$ *for some $E_{i,w}$. Then there is a nondefective plane rotation σ in Δ with* res $\sigma' < n$.

Proof. Apply 1.6 to obtain a nondefective plane rotation σ in Δ with $\sigma = E_{i,w}E_{j,v}$. Then $E'_{j,v}$ is an involution, and, hence, res $E'_{j,v} \leqslant n/2$. Hence, res $\sigma' < n$. Q.E.D.

5.3. *Let $n \geqslant 8$. If there is a nondefective plane rotation σ_1 with* res $\sigma_1' < n$ *then some nondefective plane behaves under Λ.*

Proof. Note that σ_1' is nondefective by 4.9 and res $\sigma_1' = 2$ or $n - 2$ by hypothesis and 4.10. We may assume that res $\sigma_1' = n - 2$. Thus, $V = R_1' \perp P_1'$ with P_1' a nondefective plane. By 5.1 we may obtain a Σ' in Ω with $\Sigma' P_1' \cap P_1'$ an anisotropic line. Let $\Sigma = \Lambda^{-1}(\Sigma')$, and set $\sigma_2 = \Sigma\sigma_1\Sigma^{-1}$. Then σ_2 is a nondefective plane rotation in Δ, P_1', and P_2' are nondefective planes and $P_1' \cap P_2'$ is an anisotropic line.

Now $R_1 + R_2 \subset T$ where T is a nondefective space of dimension 6. Let σ be a nondefective plane rotation whose residual space $R \subseteq T^*$. Then $\sigma \in C(\sigma_1, \sigma_2)$ by 1.10; hence, $\sigma' \in C(\sigma_1', \sigma_2')$. Hence, σ' acts like a rotation on P_1' and P_2' by 1.9. Hence, σ' leaves $P_1' \cap P_2'$ fixed. Since $P_1' \cap P_2'$ is anisotropic this means that σ' is identity on $P_1' + P_2'$ which is a ternary subspace of V. Thus, res $\sigma' = 2$ by 4.10. Q.E.D.

5.4. *If $n \geqslant 10$ then some nondefective plane behaves under Λ.*

Proof. By 5.3 it is enough to produce a nondefective plane rotation σ with res $\sigma' < n$. If V is anisotropic proceed as in step 3 of [11, 2.22] using 2.7 and 2.12 in place of 1.29 and 1.32. Thus, we may assume that V is isotropic.

Let $E_{i,w}$ be chosen. If res $E'_{i,w} < n/2$ we are done by 5.2 (this is always the case if $n \equiv 2 \bmod 4$). Assume res $E'_{i,w} = n/2$. Let V_1 be the residual space of $E'_{i,w}$. Then V_1 is also the fixed space of $E'_{i,w}$. Choose a nondefective plane R_1' with $R_1' \cap V_1 = Fx$. Let σ_1' be a nondefective plane rotation in Δ with res space $\sigma_1' = R_1'$. We claim $E'_{i,w}\sigma_1' \neq \sigma_1' E'_{i,w}$. Otherwise $E'_{i,w}$ acts like a rotation on R_1' by 1.9. But $E'_{i,w}x = x$ so $E'_{i,w} \mid R_1 = 1_{R_1}$. This would mean that $R_1 \cap V_1 = R_1$ contrary to the choice of R_1. Thus, $\sigma_1' E'_{i,w} \sigma_1'^{-1} E'_{i,w} \neq 1$ and, hence, is a plane rotation by 1.11. Let $\sigma_2 = \sigma_1 E_{i,w} \sigma_1^{-1} E_{i,w}$. Then res $\sigma_2 \leqslant 4$ and res $\sigma_2' = 2$. If σ_2' is defective apply 5.2 and 5.3 to conclude that some nondefective plane behaves under Λ^{-1}. If σ_2' is nondefective then R_2' behaves under Λ^{-1}. In either case there is a nondefective plane behaving under Λ by the remark made at the beginning of Section 5. Q.E.D.

5.5. *If $n \geqslant 6$ and σ_i are plane rotations in Δ $i = 1, 2$ with σ_i' also plane rotations then $R_1 = R_2 \Leftrightarrow R_1' = R_2'$.*

Proof. Apply 3.2 or 3.3. Q.E.D.

5.6. *If the plane R behaves under Λ and $\Sigma \in \Delta$ then ΣR behaves under*

Λ. In particular if V is isotropic and some hyperbolic plane behaves then all hyperbolic planes behave.

Proof. The first part is clear. The second follows from (7) of [5, p. 66]. Q.E.D.

5.7. *If $n \geqslant 8$ and R_1 and R_2 are nondefective planes that behave then $R_1 \cap R_2$ is a line if and only if $R_1' \cap R_2'$ is a line.*

Proof. It is enough to establish the implication one way. So we assume $R_1 \cap R_2$ is a line. Select σ_i in Δ with res space $\sigma_i = R_i$, $i = 1, 2$. Then $\sigma_1\sigma_2$ is a plane rotation by 1.11 and $\operatorname{res}(\sigma_1\sigma_2)' \leqslant 4$. If $\sigma_1\sigma_2$ is nondefective then $(\sigma_1\sigma_2)'$ is nondefective by 4.9 and a plane rotation by 4.10. If $\sigma_1\sigma_2$ is defective then $(\sigma_1\sigma_2)'$ is an involution and, hence, totally defective. But res space $(\sigma_1\sigma_2)' \subseteq R_1' + R_2'$ which is not totally defective. Hence, $(\sigma_1\sigma_2)'$ is a plane rotation, and, hence, $R_1' \cap R_2'$ is a line by 1.11. Q.E.D.

5.8. *Let R_1 and R_2 be nondefective planes with $R_1 \cap R_2$ a line. Then $R_1 \cap R_2$ is isotropic if and only if the residual space of $\sigma_1\sigma_2$ contains $R_1 \cap R_2$ for all σ_i in Δ with* res *space $\sigma_i = R_i$ $i = 1, 2$.*

Proof. Let $R_1 \cap R_2$ be the isotropic line Fx. Let σ_i be as stated, and let y_i be chosen such that y_i is isotropic and $R_i = Fx + Fy_i$ $i = 1, 2$. Then $\sigma_i x = \alpha_i x$, $\sigma_i y_i = \alpha_i^{-1} y_i$ and $\sigma_i \mid R_i^* = 1$. If $\alpha_1\alpha_2 \neq 1$ then $\sigma_1\sigma_2 x - x = (\alpha_1\alpha_2 - 1)x$, and, hence, x is in the residual space of $\sigma_1\sigma_2$. If $\alpha_1\alpha_2 = 1$ an easy computation will show that $\sigma_1\sigma_2$ is a defective plane rotation and x is in its residual space.

Now assume that $R_1 \cap R_2$ is an anisotropic line Fx. Let σ_i be in Δ with res space $\sigma_i = R_i$ $i = 1, 2$. Then $\sigma_1 = \tau_{y_1}\tau_x$ and $\sigma_2 = \tau_x\tau_{y_2}$ where $R_i = Fx + Fy_i$ $i = 1, 2$. Then $\sigma_1\sigma_2 = \tau_{y_1}\tau_{y_2}$ and res space $\sigma_1\sigma_2 = Fy_1 + Fy_2$. But $x \notin Fy_1 + Fy_2$. Q.E.D.

5.9. *Let $n \geqslant 6$. Let R_1 and R_2 be nondefective planes that behave under Λ. If $R_1' \cap R_2'$ is an anisotropic line then every nondefective plane R orthogonal to R_1 and R_2 behaves.*

Proof. Select σ in Δ with res space $\sigma = R$. Then $\sigma \in C(\sigma_1, \sigma_2)$ by 1.10. Hence, $\sigma' \in C(\sigma_1', \sigma_2')$, and σ' acts on R_i like a rotation by 1.9. Then σ' leaves $R_1' \cap R_2'$ fixed, and, since $R_1' \cap R_2'$ is anisotropic, σ' leaves $R_1' \cap R_2'$ pointwise fixed. Thus, σ' acts like the identity on a ternary subspace $R_1' + R_2'$. Now apply 4.10. Q.E.D.

5.10. *Let $n \geqslant 6$ and let R_1 and R_2 be nondefective planes that behave under Λ. Then*

$$(R_1, R_2) = 0 \Leftrightarrow (R_1', R_2') = 0$$

Proof. It is enough to establish $\Rightarrow$. Select σ_i in Δ with res space $\sigma_i = R_i$ $i = 1, 2$. Apply 1.10 and 5.5. Q.E.D.

5.11. *Let $n \geqslant 10$. Let R_1, R_2 and R_3 be nondefective planes that behave under Λ with $R_1 \cap R_2$ a line and $R_1' \cap R_2'$ an anisotropic line. Then $R_1' \cap R_2' \cap R_3'$ is a line if $R_1 \cap R_2 \cap R_3$ is a line.*

Proof. If $R_i = R_j$ $i \neq j$ apply 5.7. Thus, we may assume $R_1 \cap R_2 = R_2 \cap R_3 = R_1 \cap R_3 = l$, a line. We have two cases.

If $\dim[R_1 + R_2 + R_4] = 4$ we choose a nondefective quaternary space W with $R_1 + R_2 \subset W$ and $W = R_1 + R_2 + R_3$. This choice is possible since $R_1 + R_2$ is ternary, not totally defective, and, hence, may be embedded in two distinct quaternary nondefective spaces. Then $W^* \not\subset [R_1 + R_2 + R_3]^*$. Choose a vector x in W^* $x \notin [R_1 + R_2 + R_3]^*$. Choose y in W^* with $(x, y) \neq 0$. Then $Fx + Fy = \Pi$ is nondefective and $\Pi \subset W^*$, $\Pi \subset [R_1 + R_2 + R_3]^*$. But $W^* \subseteq [R_1 + R_2]^* = R_1^* \cap R_2^*$. Hence, $\Pi \subset R_i^*$ $i = 1, 2$ and Π behaves under Λ by 5.9. Thus Π' is a nondefective plane with $(\Pi', R_1') = 0 = (\Pi', R_2')$ by 5.10. But $\Pi \not\subset R_3^*$, and, hence, $(\Pi', R_3') \neq 0$ by 5.10. Now $R_1' \cap R_3' = l_1$ and $R_2' \cap R_3' = l_2$ by 5.7. If $l_1 = l_2$ then $R_3' = l_1 + l_2$ and $R_3' \subseteq R_1' + R_2'$. However, this means that $(R_3', \Pi') = 0$.

Now we assume that $\dim[R_1 + R_2 + R_3] = 3$. Thus, $R_1 + R_2 + R_3$ is a ternary space which is not totally defective. Hence, there is a line $\bar{l}$ in $R_1 + R_2 + R_3$, orthogonal to $R_1 + R_2 + R_3$ and $\bar{l} \neq l$. We seek a nondefective plane Π with $\Pi \cap [R_1 + R_2 + R_3] = l$. Let $l = Fx$. It is enough to find $y \notin R_1 + R_2 + R_3$ with $(x, y) \neq 0$. Choose W quaternary nondefective containing $R_1 + R_2 + R_3$. Since $l \neq \bar{l}$ we may select k in $R_1 + R_2 + R_3$ with $(x, k) \neq 0$. Let $z \in W^*$ $z = 0$. Then set $k + z = y$. So $\Pi = Fx + Fy$. Note that $\Pi + R_i + R_j$ is quaternary for $1 \leqslant i$, $j \leqslant 3$, $i \neq j$ and $\Pi \cap R_i \cap R_j = l$. Hence, we have the situation of the first part of the proof provided that Π behaves under Λ. This we now show. Note that every nondefective plane in W^* behaves by 5.9. Choose S a nondefective space of dimension 6 containing $\Pi + R_1 + R_2$ and W. Then every nondefective plane in S^* behaves under Λ by 5.9. We claim that we can select nondefective planes Π_1 and Π_2 in S^* with $\Pi_1' \cap \Pi_2'$ an anisotropic line—if so then Π behaves by 5.9.

Let R be a nondefective plane in S^*. Since $\dim S^* \geqslant 4$ we may select three nondefective planes Π_1, Π_2, and Π_3 in S^* with $\Pi_i \cap R = l_i$, $l_i \neq l_j$, $i \neq j$ and for $i \neq j, \Pi_i \cap \Pi_j = 0$, $1 \leqslant i, j \leqslant 3$. Since Π_i and R behave under Λ we have $\Pi_i' \cap \Pi_j' = 0$ for $i \neq j$ and $\Pi_i' \cap R' = l_i'$ with $l_i' \neq l_j'$ if $i \neq j$ by 5.7. Now one of the three lines l_i' must be anisotropic.

Hence, we may apply the reasoning of the first part of the proof so we have $R_i' \cap R_j' \cap \Pi' = L_k'$ where $\{i, j, k\} = \{1, 2, 3\}$. But R_i and Π are distinct, hence, so are the R_i' and Π'. Hence, $\Pi' \cap R_2' = L_3'$ and $\Pi' \cap R_2' = L_2'$. Thus, $L_2' = L_3'$. Similarly $L_1' = L_2'$. Q.E.D.

5.12. *Let $n \geqslant 10$ and let R_1 and R_2 be nondefective planes that behave under Λ. Then $R_1 \cap R_2$ is an isotropic line if and only if $R_1' \cap R_2'$ is an isotropic line.*

Proof. By 5.7 $R_1' \cap R_2'$ is a line if $R_1 \cap R_2$ is. Assume that $R_1' \cap R_2'$ is anisotropic but that $R_1 \cap R_2$ is isotropic. Choose σ_i in Δ with res space $\sigma_i = R_i$. Then $R_1 \cap R_2$ is contained in the residual space R of $\sigma_1\sigma_2$ and $R_1' \cap R_2'$ is not contained in the residual space of $(\sigma_1\sigma_2)' = R'$ by 5.8. Hence, $R_1 \cap R_2 \cap R$ is a line and $R_1' \cap R_2' \cap R' = 0$. This contradicts 5.11. Q.E.D.

5.13. *Let $n \geqslant 10$, and let R_1 be a nondefective plane that behaves under Λ. Let R be a nondefective plane orthogonal to R_1. Then R behaves under Λ.*

Proof. Working inside of R^* and applying 5.1 we obtain Σ in Δ with $\Sigma R_1 \subset R^*$, and $\Sigma R_1 \cap R_1$ an anisotropic line. Let $R_2 = \Sigma R_1$. Then R_2 behaves by 5.6. By 5.12 $R_1' \cap R_2'$ is anisotropic. Now apply 5.9. Q.E.D.

5.14. *If $n \geqslant 10$ then all nondefective planes behave under Λ.*

Proof. As in 2.27 of [11]. Q.E.D.

5.15. *If $n \geqslant 10$ and $\{R_i\}$ is a set of nondefective planes that behave under Λ then $\bigcap_i R_i$ is a line if and only if $\bigcap_i R_i'$ is a line.*

Proof. The general case follows easily from the case of three planes and 5.7. We proceed as in 5.11. Note that the assumption "$R_1' \cap R_2'$ is anisotropic" is used in the proof of 5.11 only when we use 5.9. We eliminate the assumption by appealing to 5.13 instead. Q.E.D.

5.16. *Let $n \geqslant 10$, and let R_1 and R_2 be nondefective planes with $(R_1, R_2) = 0$. Let $R \subset R_1 \perp R_2$ with R nondefective. Then $R' \subset R_1' \perp R_2'$.*

Proof. By 5.14 all nondefective planes behave. Choose nondefective planes R_i $i \geqslant 2$ with $V = \perp_{i=1}^{m} R_i$. Then $V = \perp_{i=1}^{m} R_i'$ by 5.10 and a dimension argument. Then $(R', R_i') = 0$ for $i \geqslant 3$ also by 5.10, and, hence, $R' \subset R_1' \perp R_2'$. Q.E.D.

6. The Automorphism φ_g

Let g be a semilinear automorphism of V with field automorphism u. We say that g preserves Q if and only if there is an α in $\dot{F}$ with $Q(g(x)) = \alpha(Q(x))^u\ \forall x$ in V. We say that g preserves orthogonality if and only if $(x, y) = 0$ implies $(gx, gy) = 0$. This condition is equivalent to the condition $(gx, gy) = \beta((x, y)^u)$ for some β in $\dot{F}$ and all x, y in V.

6.1. *Let g be a semilinear automorphism of V with associated field automorphism u*

(i) *if g preserves Q then g^{-1} preserves Q;*

(ii) *if g preserves Q then g preserves orthogonality—in fact if $Q(gx) = \alpha(Q(x))^u$ for all x in V then $(gx, gy) = \alpha((x, y))^u$ for all x and y in V;*

(iii) *if g preserves orthogonality, then g^{-1} preserves orthogonality;*

(iv) *g preserves Q if and only if $Q(g(x)) = \alpha(Q(x))^u$ and $(x, y) = 0$ implies $(gx, gy) = 0$ for all anisotropic x and y in V and some α in $\dot{F}$.*

Proof. The proofs of (i)–(iii) are straightforward. We verify (iv). It is clear that we may assume that V is isotropic. Let w be an isotropic vector. Choose y anisotropic with $(y, w) = 0$. Then the conditions imply that $(gy, gw) = 0$. Note that $w + y$ is anisotropic. Now compute $Q(gw) = Q(gw + gy + gy)$.

The other way is clear. Q.E.D.

Now let $\sigma \in GL_n(V)$. Then $g\sigma g^{-1}$ is in $GL_n(V)$ and $\varphi_g(\sigma) = g\sigma g^{-1}$ defines an automorphism of $GL_n(V)$ (see Section 4 of [9]). With obvious notation $\varphi_{g_1} \circ \varphi_{g_2} = \varphi_{g_1 g_2}$ and $\varphi_{g_1}^{-1} = \varphi_{g_1^{-1}}$.

6.2. *If σ and $\varphi_g(\sigma)$ are in O_n then the residual space of $\varphi_g(\sigma)$ is gR and its fixed space is gP.*

Proof. As in 3.1 of [11]. Q.E.D.

Now consider $Sp_{2m}(V)$. By 2.3 of [10] φ_g induces an automorphism of $Sp(V)$ if and only if g preserves orthogonality. In particular φ_g induces an automorphism of $Sp(V)$ if g preserves Q.

Now consider a symplectic transvection $\tau = \tau_{a,\lambda}$ where $\tau_{a,\lambda}(x) = x + \lambda(x, a), a, \lambda \in \dot{F}$. Then τ is an orthogonal transvection if and only if $\lambda = Q(a)^{-1}$. Suppose that g preserves orthogonality; then $\varphi_g(\tau_{a,\lambda}) = \tau_{ga,\lambda}\, u_\alpha - 1$ by 2.4 of [10]. In particular if $\tau_{a,\lambda}$ is an orthogonal transvection then $\varphi_g(\tau_{a,\lambda})$ will also be orthogonal if and only if $Q(g(a)) = \alpha(Q(a))^u$. Thus, if g preserves orthogonality φ_g maps orthogonal transvections onto orthogonal transvections if and only if g preserves Q.

6.3. *Let $n \geqslant 4$ and assume that g preserves Q. Then φ_g induces an automorphism of O, O^+, O', and Ω.*

Proof. φ_g induces an automorphism of Sp by [10, 2.3] and 6.1 (ii). By the remarks immediately preceeding this proposition φ_g maps orthogonal transvections to orthogonal transvections; hence, $\varphi_g(O) \subseteq O$. Applying the same argument to g^{-1} using the relation $\varphi_{g^{-1}} = \varphi_g^{-1}$ and 6.1 (i) we get $\varphi_g(O) = O$. Similarly we can conclude $\varphi_g O^+ = O^+$ provided that $\varphi_g O^+ \subseteq O^+$. But this later inclusion follows since O^+ is characterized by the parity of the number of orthogonal transvections used in generation.

Now let $\sigma \in O'$. Express $\sigma = \prod_{i=1}^{2r} \tau_{a_i}$. Then $\prod_{i=1}^{2r} Q(a_i) = \delta^2$, $\delta \in \dot{F}$. But $\varphi_g(\tau_{a_i}) = \tau_{g(a_i)}$ since g preserves Q. Hence, $\varphi_g(\sigma) = \prod_{i=1}^{2r} \tau_{(ga_i)}$ and $\prod_{i=1}^{2r} Q(ga_i) = \alpha^{2r}(\delta^2)^u$. Since $uF^2 = F^2$ we have $\varphi_g(\sigma) \in O'$. As above this is enough to conclude $\varphi_g O' = O'$.

The relation $\varphi_g \Omega = \Omega$ is obvious. Q.E.D.

6.4. *If $n \geqslant 8$ and φ_g induces an automorphism of Δ then g preserves Q.*

Proof. We show first that g preserves orthogonality. Let x_1 and x_2 be independent vectors with $(x_1, x_2) = 0$. It is possible to choose nondefective planes R_1 and R_2 such that $x_i \in R_i$ and $(R_1, R_2) = 0$ $i = 1, 2$. Let $\sigma_i \in \Delta$ with res space $\sigma_i = R_i$ $i = 1, 2$. Then σ_1 and σ_2 permute by 1.8 (ii), hence, $\varphi_g(\sigma_1)$ and $\varphi_g(\sigma_2)$ permute. But $\varphi_g(\sigma_1)$ and $\varphi_g(\sigma_2)$ have residual spaces gR_1 and gR_2, respectively, by 6.2 and these spaces are nondefective by 4.9. Since $gR_1 \cap gR_2 = 0$ we have $(gR_1, gR_2) = 0$ by 1.8 (iii). Hence, $(gx_1, gx_2) = 0$. Thus, φ_g induces an automorphism of $Sp(V)$ which maps symplectic transvections to symplectic transvections by 2.3 of [10]. We now show that φ_g maps orthogonal transvections to orthogonal transvections. Let τ_a be an orthogonal transvection. Then there is an orthogonal transvection τ_b such that $\tau_a\tau_b \in \Delta$ and $\tau_a\tau_b$ is a nondefective plane rotation. Hence, $\varphi_g(\tau_a\tau_b) \in \Delta$. Then $\varphi_g(\tau_a) \in O$ if and only if $\varphi_g(\tau_b) \in O$. To show that $\varphi_g(\tau_a)$ is in O it is then enough to show that the product of two distinct symplectic transvections is in O^+ if and only if each is in O. So let $\tau_{a,\lambda}\tau_{b,u} \in O$. Then $\tau_{a,\lambda}\tau_{b,u}(x) = x + \lambda(a, x)a + u(x, b)(b + \lambda(a, b)a)$. If $Fa^* \neq Fb^*$ choose x with $(x, a) = 1$ and $(x, b) = 0$. Then $\tau_{a,\lambda}\tau_{b,u}(x) = x + \lambda a$. But $Q(x + \lambda a) = Q(x)$, and, hence, $\lambda = Q(a)^{-1}$. Hence, $\tau_{a,\lambda} \in O$, and, thus, $\tau_{b,u} \in O$. If $Fa^* = Fb^*$ then $Fa = Fb$. Hence, $b = \zeta a$ and $\tau_{b,u} = \tau_{\zeta a,u}$. But $\tau_{\zeta a,u} = \tau_{a,\zeta^2 u}$ by Section 1A of [10]. Hence, $\tau_{a,\lambda}\tau_{b,u} = \tau_{a,\lambda}\tau_{a\zeta^2 u} = \tau_{a,\lambda+\zeta^2 u} \in O^+$. This is possible if and only if $\tau_{a,\lambda+\zeta^2 u} = 1$ in which case $\tau_{a,\lambda}$ and $\tau_{b,u}$ are not distinct.

Now apply the remarks immediately preceeding 6.3. Q.E.D.

6.5. *Let $n \geqslant 4$ and let g_1 and g_2 be semilinear automorphisms of V that preserve Q; then φ_{g_1} and φ_{g_2} induce automorphisms on O, O^+ and Δ and $\varphi_{g_1} = \varphi_{g_2}$ if and only if $g_1 = \alpha g_2$ some α in $\dot{F}$.*

Proof. Clear. Q.E.D.

7. The Automorphisms of Δ

In this section we generally assume that $n \geqslant 10$. Then all nondefective planes in V behave under Λ by 5.14. We establish a bijection of the set of nondefective planes onto itself in the following manner. Let R be a nondefective plane, and let σ in Δ be chosen with res space $\sigma = R$. Then R' is nondefective by 4.9. Let R correspond to R'. This map does not depend on the σ chosen by the remark made at the beginning of Section 5; it is injective by 5.5; the inverse of this correspondence is the similar correspondence induced by Λ^{-1}, and, hence, it is surjective. By 5.10 we see that $(R_1', R_2') = 0$ if and only if $(R_1, R_2) = 0$.

7.1. *Let V be any quadratic space to which the above may be applied* (e.g. *if $n \geqslant 10$*). *If $R = R'$ for all nondefective planes R then $\Lambda = 1$.*

Proof. As in 4.5 of [11]. Q.E.D.

7.2. *Let $n \geqslant 4$. Let l_1 and l_2 be distinct lines in V. Then there are nondefective planes R_1 and R_2 with $l_i \subseteq R_i$ $i = 1, 2$ and $R_1 \cap R_2 = 0$. Moreover if l_1 and l_2 are orthogonal, R_1 and R_2 may be chosen to be orthogonal.*

Proof. If $(l_1, l_2) = 0$ then $l_2 \subseteq l_1{}^*$. Choose x in $l_1{}^*$ with $(x, l_2) \neq 0$. Let $l_2 = Fy$, and set $R_2 = Fx + Fy$. The rest is clear.

So assume $(l_1, l_2) \neq 0$. Let $l_i = Fx_i$ $i = 1, 2$. Then $Fx_1 + Fx_2$ is a nondefective plane. Choose z_i in $[Fx_1 + Fx_2]^*$ with $(z_1, z_2) \neq 0$. Let $R_i = Fx_i + F(y_j + z_i)$ $i \neq j$ $i, j = 1, 2$. Q.E.D.

7.3. *Let $n \geqslant 6$ and let l be a line not contained in the plane Π. Then there is a nondefective plane R with $l \subset R$ and $R \cap \Pi = 0$.*

Proof. First assume Π is nondefective. If $(l, \Pi) = 0$ it is clear. So assume $(l, \Pi) \neq 0$. Let $l = Fx$. Now $V = \Pi \perp \Pi^*$ and $x = y + z$ $y \in \Pi$, $z \in \Pi^*$ and $y \neq 0$, $z \neq 0$. Choose w in Π^* with $(z, w) \neq 0$, and set $R = Fx + Fw$. Clearly $\Pi \cap R = 0$.

Now we assume that Π is defective. Choose a quaternary nondefective space T with $\Pi \subset T$. If $(l, T) = 0$ the result is clear. So assume $(l, T) \neq 0$.

Let $l = Fx$ and express $x = y + z$ where $y \in T$, $z \in T^*$. Then $y \neq 0$ and $(x, y) \neq 0$. If $z \neq 0$ choose w in T^* with $(w, z) \neq 0$, and let $R = Fx + Fw$. If $z = 0$ then $x \in T$. Choose t in T with $(x, t) \neq 0$. Choose w in T^* $w \neq 0$. Set $R = Fx + F(t + w)$. Q.E.D.

The correspondence of nondefective planes which we have established above induces a correspondence of lines in the following manner. Let l be a line in V, and let R_i be the set of nondefective planes in V which contain l. Thus, $l = \bigcap_i R_i$. Then $\bigcap R_i'$ is a line l' by 5.15 and l' is isotropic if and only if l is isotropic by 5.12. This correspondence is injective by 7.2 and 5.7. The similar correspondence induced by Λ^{-1} is easily seen to be the inverse of the one induced by Λ and hence the above mapping is a bijection of lines of V.

7.4. *Let $n \geqslant 10$ and let R_1 and R_2 be nondefective planes in V and let l, l_1, and l_2 be three lines in V with l_1 and l_2 distinct. Then*

(i) $(l_1, l_2) = 0 \Leftrightarrow (l_1', l_2') = 0$,

(ii) $l = R_1 \cap R_2 \Leftrightarrow l' = R_1' \cap R_2'$,

(iii) $l \subset l_1 + l_2 \Leftrightarrow l' \subset l_1' + l_2'$.

Proof. In all cases it is enough to establish the implication one way

(i) follows from 7.2 and 5.10 and the way the correspondence of lines is defined, (ii) follows from 5.7 and the way the bijection of lines is defined.

(iii) if $(l_1, l_2) \neq 0$ then $l_1 + l_2$ is a nondefective plane R and l', l_1', and l_2' are in R'. But $R' = l_1' + l_2'$.

If $(l_1, l_2) = 0$ we have $(l_1', l_2') = 0$ by (i). The result is clearly true if $l = l_1$ or l_2 so assume l, l_1, and l_2 are distinct. Then $(l, l_i) = 0$ $i = 1, 2$, and, hence, by (1) $(l_1', l_i') = 0$. Apply 7.2 to choose R_1 and R_2 nondefective with $l_i \subset R_i$ and $(R_1, R_2) = 0$ $i = 1, 2$. Then we may choose R nondefective with $R \subset R_1 \perp R_2$ and $l \subset R$. By 5.16 $R' \subset R_1' \perp R_2'$.

Now assume $l' \not\subset l_1' \perp l_2'$. Then $l' + l_1' + l_2'$ is a totally defective ternary subspace. But $l' \subset R' \subset R_1' \perp R_2'$. Hence, $R_1' \perp R_2'$ contains a ternary totally defective subspace. This is absurd since $R_1' \perp R_2'$ is quaternary nondefective. Q.E.D.

Now consider out bijection of lines $l \rightarrow l'$. In light of 7.4 (iii) we may apply the Fundamental Theorem of Projective Geometry to obtain a semilinear automorphism g of V with $gl = l'$. Then g preserves orthogonality by 7.4 (i) and g maps anisotropic vectors onto anisotropic vectors by

5.12. Since g preserves orthogonality, φ_g induces an automorphism of $S_p(V)$. Now φ_g restricted to $O(V)$ will induce an automorphism of $O(V)$ if and only if g preserves Q. We now show that the conditions we have on g will insure this.

7.5. *Let g be a semilinear automorphism of V. Then g preserves Q if and only if g preserves orthogonality and $Q(gx) = 0$ whenever $Q(x) = 0$.*

Proof. If g preserves Q we have shown that g preserves orthogonality. The other condition is obvious from the equation $Q(gx) = \alpha Q(x)^u$.

We now assume that g preserves orthogonality and $Q(gx) = 0$ whenever $Q(x) = 0$. If x is anisotropic there is a unique nonzero scalar α_x such that $Q(gx) = \alpha_x Q(x)^u$. If x is isotropic we set $\alpha_x = \alpha$. We must show that $\alpha_x = \alpha$ for all x (the "α" mentioned is the unique α with $(gx, gy) = \alpha(x, y)^u$).

First, we assume that V is isotropic. Express $V = H \perp H^*$ where H is a hyperbolic plane. It is easy to see that $\alpha_x = \alpha$ for all x in H. Let $y \in H^*$. Since H is universal we may choose x in H with $Q(x) = Q(y)$. We show that $\alpha_y = \alpha_x = \alpha$. Clearly we may assume that y is anisotropic. Now $x + y$ is isotropic so $\alpha_{x+y} = \alpha$. Thus, $Q(g(x + y)) = \alpha(Q(x + y))^u$. Since $g(x + y)$ is also isotropic and since g preserves orthogonality we have $0 = Q(gx + gy) = Q(gx) + Q(gy) = \alpha Q(x)^u + \alpha_y Q(y)^u$. The rest is clear.

We now assume that V is anisotropic. Let H be a hyperbolic plane over F, and let $\bar{V} = V \perp H$. It is an easy matter to construct a semilinear isomorphism h of H with respect to u such that $Q(x)^u = Q(h(x))$ for all x in H. Let $\bar{g}$ be the semilinear isomorphism of $\bar{V}$ defined by $\bar{g} = g \perp h$. It is easy to check that $\bar{g}$ preserves orthogonality and $Q(\bar{g}(x)) = 0$ whenever $Q(x) = 0$, x in $\bar{V}$. Now apply the results of the first part of the proof.

Q.E.D.

Thus, we have the following situation. The automorphism Λ of Δ produced a bijection of lines of V, $l \to l'$ and a semilinear automorphism g of V such that $gl = l'$. This semilinear map yields an automorphism φ_g of Δ. Hence, $\varphi_g^{-1}\Lambda$ is an automorphism of Δ. This automorphism produces a bijection of the nondefective planes of V which is identity. Hence, by 7.1 $\varphi_g^{-1}\Lambda = 1$. We collect our results as follows.

7.6. *Let V be a nondefective quadratic space of dimension at least* 10 *over a field of characteristic* 2 *having at least four elements. Let Δ be one of the groups $\Omega(V)$ or $O'(V)$. Then Λ is an automorphism of Δ if and only if $\Lambda = \varphi_g \mid \Delta$ where g is a semilinear isomorphism of V and $Q(gx) = \alpha Q(x)^u$ some α in $\dot{F}$, all x in V.*

8. The Automorphisms of $O(V)$ and $O^+(V)$

Now let $\bar{\Delta}$ denote one of the two groups, $O(V)$ or $O^+(V)$, V as above. Let $\bar{\Lambda}$ be an automorphism of $\bar{\Delta}$. Since $\Omega(V)$ is also the commutator subgroup of $O^+(V)$, $\bar{\Lambda}$ induces an automorphism of $\Omega(V)$ and $\bar{\Lambda} \mid \Omega(V)$ is $\Phi_g \mid \Omega(V)$ by 7.6. We will show that $\bar{\Lambda}$ is $\Phi_g \mid \bar{\Delta}$.

Let $\bar{\Lambda} \mid \Omega(V)$ be denoted by Λ. Then given $\bar{\Lambda}$ we obtain Λ and, hence, a bijection of the nondefective planes of V as in Section 7. We claim that $\bar{\Lambda}$ is the identity automorphism if and only if the associated correspondence of nondefective planes is the identity.

8.1. *Let $\bar{\Lambda}$ be an automorphism of $\bar{\Delta}$. Then $\bar{\Lambda}$ is the identity automorphism if and only if the bijection of nondefective planes which it induces is the identity.*

Proof. Let Σ be in $\bar{\Delta}$ and let R be any nondefective plane in V. Apply 1.3a to obtain σ in $\Omega(V)$ such that res space σ is R. If the bijection of nondefective planes is the identity then res space $\bar{\Lambda}\sigma$ is R. Now consider $\Sigma\sigma\Sigma^{-1}$. This is a nondefective plane rotation in $\Omega(V)$ with res space ΣR and the res space of $\bar{\Lambda}(\Sigma\sigma\Sigma^{-1})$ is also ΣR. But the res space of $\bar{\Lambda}(\Sigma\sigma\Sigma^{-1})$ is the res space of $(\bar{\Lambda}\Sigma)(\bar{\Lambda}\sigma)(\bar{\Lambda}\Sigma^{-1})$ which is $(\bar{\Lambda}\Sigma)R$. Hence, $(\bar{\Lambda}\Sigma)R = \Sigma R$ for all nondefective planes R and $\Sigma^{-1}(\bar{\Lambda}\Sigma)l = l$ for all lines in V. Now apply 2.6 to conclude that $\bar{\Lambda}\Sigma = \Sigma$ for all Σ in $\bar{\Delta}$.

The converse is obvious. Q.E.D.

8.2. Theorem. *Let $\bar{\Lambda}$ be an automorphism of $\bar{\Delta}$. Then $\bar{\Lambda} = \Phi_g \mid \bar{\Delta}$ where g is a semilinear automorphism of V with respect to underlying field automorphism u and g preserves the quadratic structure of V. Conversely, any such Φ_g induces an automorphism of $\bar{\Delta}$.*

Proof. The first part follows by applying 7.6 and 8.1 The converse follows from 6.3. Q.E.D.

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